2013

On well-quasi-orderings

Forrest Thurman
University of Central Florida

ON WELL-QUASI-ORDERINGS

by

FORREST B. THURMAN

A thesis submitted in partial fulfillment of the requirements
for the Honors in the Major Program in Mathematics
in the College of Sciences
and in The Burnett Honors College
at the University of Central Florida
Orlando, Florida

Spring Term 2013

Thesis Chair: Dr. Joseph Brennan

ABSTRACT

A quasi-order is a relation on a set which is both reflexive and transitive, while a *well*-quasi-order has the additional property that there exist no infinite strictly descending chains nor infinite antichains. Well-quasi-orderings have many interesting applications to a variety of areas which includes the strength of certain logical systems, the termination of algorithms, and the classification of sets of graphs in terms of excluded minors. My thesis explores how well-quasi-orderings are related to these topics through examples of four known well-quasi-orderings which are given by Dickson's Lemma, Higman's Lemma, Kruskal's Tree Theorem, and the Robertson-Seymour Theorem. The well-quasi-ordering conjecture for matroids is also discussed, and an original proof of Higman's Lemma is presented.

ACKNOWLEDGMENTS

I would like to thank each of my committee members Joseph Brennan, Michael Reid, and Donald Jones for your invaluable encouragement and advice throughout the thesis writing process.

TABLE OF CONTENTS

1 INTRODUCTION

The notion of a well-quasi-order is simple to define, yet it proves to be quite a powerful concept. Well-quasi-orderings have been used to study the strengths of logical systems, to prove the termination of certain algorithms, and to show that minor-closed sets of graphs can be characterized by a finite set of excluded minors. That well-quasi-orderings have such numerous applications is suggested by Theorem 3.2 which gives 7 equivalent conditions for when a quasi-order is a well-quasi-order. These rather simple to prove equivalencies have important consequence when considering particular well-quasi-orders.

In section 2, three important theorems in infinite combinatorics are stated and proven. These results are used freely throughout. In section 3, the definitions of a quasi-order and well-quasi-order are given along with a proof of Theorem 3.2 and other basic properties of quasi-orders and well-quasi-orders.

Each of the following sections highlights a particular well-quasi-order along with a different application:

- Section 4: Dickson's Lemma concerns the well-quasi-ordering of finite products of well-quasi-orders. We present an application of Dickson's Lemma to proving the termination of Buchberger's algorithm.

- Section 5: Higman's Lemma covers the well-quasi-ordering of finite strings with labels from a well-quasi-order. The proof given is original and makes use of an inductive property of well-quasi-orderings.

- Section 6: Kruskal's Tree Theorem concerns the well-quasi-ordering of finite trees with labels from a well-quasi-order. We discuss Friedman's Finite Form of Kruskal's Tree Theorem and its application to mathematical logic.

- Section 7: The Robertson-Seymour Theorem covers the well-quasi-ordering of finite graphs. We discuss its application to characterizing minor-closed families of graphs.

- Section 8: Well-Quasi-Ordering Conjecture for F-Representable Matroids details a conjecture for F-Representable Matroids which is analogous to the Robertson-Seymour Theorem.

2 PRELIMINARIES

Here we will discuss three important theorems in infinite combinatorics which we will apply to our study of well-quasi-orders. Each of these theorems concerns the unavoidable existence of certain structures.

Notation and Definitions: We will use $\mathbb{N}$ to denote the set of natural numbers, while $[n]$ will denote the finite set $\{1, 2, ..., n\}$. For any infinite subset $M \subseteq \mathbb{N}$ we will use $[M]^2$ to denote the set $\{(i, j) : i, j \in M \text{ and } i < j\}$. A rooted tree has a distinguished vertex v_0, called the root vertex. Every vertex of a rooted tree has a set of children. A finite path in a rooted tree a sequence $v_0, v_1, \ldots, v_n$ such that v_i is a child of v_{i-1} for $1 \leq i \leq n$. An infinite path is an infinite sequence $v_0, v_1, \ldots$ such that v_i is a child of v_{i-1} for $i > 0$. For vertices v_1, v_2 of a rooted tree, v_2 is an ancestor of v_1 if there exists a path in the tree that begins with v_1 and ends with v_2.

Theorem 2.1. *(Infinite Pigeon Hole Principle) Let $f : X \to [n]$ with X an infinite set. Then there exists $Y \subseteq X$ such that Y is infinite and $f(Y)$ is a singleton.*

Proof. There must exist $k \in [n]$ such that $f^{-1}(k)$ is infinite. Otherwise, $\mathbb{N}$ would be a finite union of finite sets and so would be finite, which is a contradiction. We can take Y to be $f^{-1}(k)$. $\square$

Definition. *For $f : X \to [n]$, if $f(Y)$ is a singleton, We call Y an f-homogeneous subset of X.*

Theorem 2.2. *(Infinite Ramsey's Theorem) Let $f : [\mathbb{N}]^2 \to [n]$. Then $\exists M \subseteq \mathbb{N}$ such that M is infinite and $f([M]^2)$ is a singleton.*

Proof. Let $A_1 = \mathbb{N}$ and $a_1 = 1$. Assume A_i and a_i are defined. We define a function h_i from the infinite set $\{j \in A_i \text{ such that } a_i < j\}$ to $[n]$ by $h_i(j) = f(a_i, j)$. By the infinite pigeonhole principle there exists an h_i-homogeneous subset H_i. Let a_{i+1} be the smallest element of H_i and take A_{i+1} to be $\{a_1, a_2, \ldots, a_i\} \bigcup H_i$. Let $A = \bigcap_i A_i$. Note that $A = \{a_1, a_2, \ldots\}$. Note also that by the construction of the A_i's the value of $f(a_i, a_j)$ for $i < j$ depends only on a_i. Define a function $g : A \to [n]$ by $g(a_i) = f(a_i, a_{i+1})$. By the pigeonhole principle, there exists a g-homogeneous subset M. Then $f(M)$ is a singleton. □

Theorem 2.3. *(König's Lemma) Let T be a rooted tree in which each vertex has finitely many children. If T has paths of arbitrarily long finite length, then T contains a path of infinite length.*

Proof. For each $n \in \mathbb{N}$, we will construct a path $P_n = a_0, a_1, \ldots, a_n$ such that P_{n+1} extends P_n. This guarantees that the path $P_\infty = a_0, a_1, a_2, \ldots$ will be infinite. Let $P_0 = a_0$ be the root of T, and let $\varphi(P_n)$ be the statement "There exists paths of arbitrarily long finite length extending P_n". We will show that $\varphi(P_n) \Rightarrow \exists a_{n+1}$ such that $\varphi(P_{n+1})$ where P_{n+1} is the path $a_0, a_1, \ldots, a_n, a_{n+1}$. Since $\varphi(P_0)$ is an assumption of the lemma, this will guarantee that P_0 can be extended to some P_1 such that $\varphi(P_1)$, which in turn can be extended to some P_2 such that $\varphi(P_2)$, and so on. Assume $\varphi(P_n)$ for $P_n = a_0, a_1, \ldots, a_n$. This means that there are paths of arbitrarily long finite length extending P_n. Let $B = \{b_1, b_2, \ldots, b_k\}$ be the finite set of children of a_{n+1}. If P_n could not be extended to some P_{n+1} such that $\varphi(P_{n+1})$, then for each child b_i, there would exist some number m_i such that the length of any path extending $a_0, a_1, \ldots, a_n, b_i$ would be bounded by m_i. Since each path of P_n must be extended with a child of a_n, we see that $M = \max(\{m_i | 1 \leq i \leq k\})$ would be an upper bound for the length of any path extending $P_n = a_0, a_1, \ldots, a_n$. This contradicts the fact that there are paths of arbitrary length extending P_n. Thus, there is some b_i such that φ holds for $P_{n+1} = a_0, a_1, \ldots, a_n, b_i$. □

3 GENERAL QUASI-ORDERS AND WELL-QUASI-ORDERS

Definition. *A **quasi-order** is a pair $(Q, \preceq)$, where Q is a set and $\preceq$ is a binary relation over Q which is both reflexive: $\forall x \in Q, x \preceq x$, and transitive: $\forall x, y, z \in Q$, we have $(x \preceq y \wedge y \preceq z) \Rightarrow x \preceq z$.*

We will use $x \prec y$ to mean $x \preceq y$ and $y \npreceq x$. Note that, for a quasi-order this is different than saying $x \preceq y$ and $x \neq y$.

Definition. *A **partial order**, is a quasi-order, Q, which is antisymmetric: $\forall x, y \in Q, (x \preceq y \wedge y \preceq x) \Rightarrow x = y$.*

Definition. *A **total order**, is a partial order $(Q, \preceq)$ which is total: $\forall x, y \in Q, x \preceq y \vee y \preceq x$.*

For a quasi-order, Q, we write $x \equiv y$ if $x \preceq y$ and $y \preceq x$. For $x \in Q$ we will use $[x]$ to denote the equivalence class of x, and $Q/\equiv$ to denote $\{[x] | x \in Q\}$, the set of equivalence classes under $\equiv$. The set $Q/\equiv$ can be partially ordered by defining $[x] \preceq_{\equiv} [y]$ if $x \preceq y$. It is simple to verify that this definition is well defined, and that $(Q/\equiv, \preceq_{\equiv})$ is a partial order.

Definition. *A sequence $\{a_n\} \subseteq Q$ is called **increasing(decreasing)** if $a_i \preceq a_j (a_i \succeq a_j)$ for all $i < j$. Similarly a sequence $\{a_n\} \subseteq Q$ is called **strictly increasing(strictly decreasing)** if $a_i \prec a_j (a_i \succ a_j)$ for all $i < j$. A set $X \subset Q$ is called an **antichain** if for any $x_1, x_2 \in X$, $x_1 \neq x_2$ implies x_1 and x_2 are incomparable. A set $X \subset Q$ is called a **chain** if for any $x_1, x_2 \in X$, either $x_1 \preceq x_2$ or $x_2 \preceq x_1$. An element x of Q is called **minimal** if there does not exist $y \in Q$ such that $y \prec x$.*

We call $X \subseteq Q$ *open* if $x \in X$ and $a \prec x$ implies $a \in X$. Similarly, We call $X \subseteq Q$

closed if $x \in X$ and $a \succ x$ implies $a \in X$. For $X \subseteq Q$ we define the closure of X, $\mathrm{cl}(X)$ by $\mathrm{cl}(X) = \{q \in Q | \exists x \in X \text{ such that } x \leq q\}$.

Theorem 3.1. *Let X,$Y \subseteq Q$ with $(Q, \prec)$ a quasi-order. Then we have:*

1. $X \subseteq \mathrm{cl}(X)$
2. $\mathrm{cl}(X) = \mathrm{cl}(\mathrm{cl}(X))$
3. $X \subseteq Y$ *implies* $\mathrm{cl}(X) \subseteq \mathrm{cl}(Y)$.
4. X *is closed* $\Rightarrow \mathrm{cl}(X) = X$.
5. $\bigcup_{i \in I} X_i$ *is closed(open) provided every* X_i *is closed(open).*
6. $\bigcap_{i \in I} X_i$ *is closed(open) provided every* X_i *is closed(open).*

Proof.

1. The relation $\preceq$ is reflexive so $x \preceq x$. Thus, $x \in X \Rightarrow x \in \mathrm{cl}(X)$.
2. By (1), we have $\mathrm{cl}(X) \subseteq \mathrm{cl}(\mathrm{cl}(X))$. Let $x \in \mathrm{cl}(\mathrm{cl}(X))$. Then there exists $a \in \mathrm{cl}(X)$ such that $a \preceq x$. Since $a \in \mathrm{cl}(X)$, there exists $b \in X$ such that $b \prec a$. By transitivity of $\preceq$, we have $b \preceq x$. Thus, $x \in \mathrm{cl}(X)$.
3. Let $X \subset Y$ and $a \in \mathrm{cl}(X)$. Then there exists $x \in X$ such that $x \preceq a$. Since $x \in Y$ also, we have $a \in \mathrm{cl}(Y)$.
4. Suppose X is closed. By (i), we have $X \subseteq \mathrm{cl}(X)$. To show $\mathrm{cl}(X) \subseteq X$. Let $a \in \mathrm{cl}(X)$. Then there exists $x \in X$ such that $x \preceq a$. Since X is closed, we have $a \in X$.
5. We will prove the statement for closed sets. The proof for open sets is similar. Let $X = \bigcup_{i \in I} X_i$ where each X_i is closed. We must show that if $x \in X$ and $x \preceq a$ then $a \in X$. So suppose $x \in X$ and $x \preceq a$. Then $x \in X_i$ for some i. Since X_i is closed, we have $a \in X_i$ and hence $a \in X$.

6. Again, we prove the statement for closed sets. The proof for open sets is similar. Let $X = \bigcap_{i \in I} X_i$ with X_i closed for all i. We must show that if $x \in X$ and $x \preceq a$ then $a \in X$. So suppose $x \in X$ and $x \preceq a$. Then $x \in X_i$ for all i. Since each X_i is closed, $a \in X_i$ for all i. Hence, $a \in X$.

□

Theorem 3.2 lists 7 equivalent properties a quasi-order may have.

Definition. *A **well-quasi-order** is a quasi-order that satisfies any of the equivalent conditions of Theorem 3.2.*

We will call condition (i) the increasing subsequence condition. Higman in [Higman, 1952] named condition (v) the finite basis property, while condition (vi) is known as the Noetherian property. Condition (iii) is often used to show that an algorithm will terminate. We will discuss how conditions (iii) and (vii) lead to different induction schemes for a well-quasi-order.

Theorem 3.2. *For a quasi-order, $(Q, \prec)$, the following conditions are equivalent:*

(i)For any infinite sequence $\{a_n\} \subseteq Q$ there exists an infinite subsequence $\{a_{n_k}\} \subseteq \{a_n\}$ such that $a_{n_i} \preceq a_{n_j}$ for all $i \leq j$.

(ii)For any infinite sequence $\{a_n\} \subseteq Q$ there exists $i < j$ such that $a_i \preceq a_j$

(iii)In Q, there are no infinite strictly decreasing sequences, nor infinite antichains.

(iv)Every nonempty subset $X \subseteq (Q/\equiv, \preceq_{\equiv})$ has a finite, non-zero number of minimal elements.

(v)Every closed set $X \subseteq Q$ is the closure of a finite set.

(vi)There exists no infinite strictly ascending sequence of closed subsets of Q.

(vii)There exists no infinite strictly descending sequence of open subsets of Q.

Proof. (i) $\Rightarrow$ (ii), (ii) $\Rightarrow$ (iii): The proof is trivial.

(iii) $\Rightarrow$ (i): We use the infinite version of Ramsey's theorem. Let $\{a_n\} \subseteq Q$ be an arbitrary infinite sequence. We can define a function $f : [\mathbb{N}]^2 \to [3]$ by $f(i,j) = 1$ if $a_i \preceq a_j$, $f(i,j) = 2$ if $a_i \succ a_j$, and $f(i,j) = 3$ if a_i and a_j are incomparable. By the infinite version of Ramsey's theorem, there exists an f-homogeneous subset M. M is the set of indices of either an increasing sequence, a strictly decreasing sequence, or a sequence which has pairwise incomparable elements. Assuming condition (iii), M must index an increasing subsequence of a_n. Since $\{a_n\}$ was arbitrary, condition (i) holds. Thus, (iii) implies (i).

(iii) $\Rightarrow$ (iv): Let $X \subseteq Q/\equiv$ be nonempty. If X had no minimal elements, then we could form an infinite strictly decreasing sequence $x_1 \succ x_2 \succ x_3 \succ \cdots$ from Q as follows. Choose $[x_1] \in X$ since X is nonempty. Assume $[x_i]$ is defined. Since $[x_i]$ is non minimal, choose $[x_{i+1}]$ such that $[x_i] \succ_\equiv [x_{i+1}]$. Thus, we have $[x_1] \succ_\equiv [x_2] \succ_\equiv [x_3] \succ_\equiv \cdots$, from which it follows that $x_1 \succ x_2 \succ x_3 \succ \cdots$. This is a contradiction. Thus, X contains a minimal element. To show that X contains only finitely many minimal elements, we can choose exactly one element x_α from each equivalence class $[x_\alpha]$ that is minimal with respect to $\preceq_\equiv$. Then the set $x_\alpha, \alpha \in A$ is an antichain, and so must be finite. This implies that the set of minimal elements of X, i.e. $\{[x_\alpha] | [x_\alpha] \text{ is minimal}\}$, is also finite.

(iv) $\Rightarrow$ (v): Let $X \subseteq Q$ be closed. If $X = \emptyset$, then X is the closure of $\emptyset$ which is finite. So assume, $X \neq \emptyset$. Then $X/\equiv$ has a finite , non-zero number of minimal elements, say $[x_1], [x_2], \ldots, [x_n]$. Let $a \in X$ be arbitrary, and consider the open set $B = \{[x] \in X/\equiv | [x] \prec_\equiv [a]\}$. Since B is nonempty, B contains a minimal element $[b]$. Since B is open, any minimal element of B is a minimal element of X. Therefore, $[b] = [x_i] \preceq_\equiv$ for some $1 \leq i \leq n$. This implies $x_i \preceq a$. Since $a \in X$ was arbitrary, we have that every $a \in X$, there is some i such that $x_i \preceq a$. Thus, we have $X \subseteq \mathrm{cl}(\{x_1, x_2, \ldots, x_n\})$. Since X is closed, we have equality $X = \mathrm{cl}(\{x_1, x_2, \ldots, x_n\})$. This shows X is the closure of a finite set.

(v) $\Rightarrow$ (vi): Assume that $A_1 \subset A_2 \subset A_3 \subset \cdots$ is a strictly increasing sequence of closed subsets of Q. Let $A = \bigcup_i A_i$. Since A is a union of closed sets, A must be closed. By condition (iv), A is generated by some finite set $X = \{x_1, x_2, \ldots, x_n\}$. Each x_i is the member of some set in $\{A_n\}$. Since the A_i's are nested, there is some i such that $X \subseteq A_i$. However, $A = \mathrm{cl}(X)$,$X \subseteq A_i$, and A_i is closed. Putting this together, we have $A = \mathrm{cl}(X) \subseteq \mathrm{cl}(A_i) = A_i \subseteq A$. Thus, $A_i = A$ which contradicts $\{A_n\}$ being a strictly ascending sequence. Thus (iv) implies (v).

(vi) $\Rightarrow$ (ii): Let $\{a_n\}$ be a sequence for which $i < j \Rightarrow a_i \not\preceq a_j$. Then $\mathrm{cl}(\{a_1\} \subset \mathrm{cl}(\{a_1, a_2\}) \subset \mathrm{cl}(\{a_1, a_2, a_3\}) \ldots$ would be an infinite strictly ascending sequence of closed sets.

(vi) $\Leftrightarrow$ (vii): If there exists a strictly increasing sequence of closed sets $A_0 \subset A_1 \subset A_2 \subset \cdots$, then $A_0^C \supset A_1^C \supset A_2^C \supset \cdots$ is a strictly decreasing sequence of open sets and vice versa. □

Condition (iii) shows one can induct over the elements of Q using this induction scheme.

Theorem 3.3. *Let φ be a statement for which*

1. *$\varphi(x)$ for all minimal x.*

2. *For all $x \in Q$, φ holds for x provided φ holds for all $y \prec x$.*

Then $\forall x \in Q, \varphi(x)$

This induction scheme is valid.

Proof. Suppose conditions 1. and 2. hold, and suppose to the contrary that there is an $x_1, \in Q$ for which φ did not hold. Then x_1 can not be minimal and there must exist x_2 such that $x_2 \prec x_1$ and φ does not hold for x_2. We can repeat this argument ad infinitum to construct an infinite strictly decreasing subsequence $x_1 \succ x_2 \succ x_3 \succ \cdots$, which contradicts the fact that Q is a well-quasi-order. $\square$

In light of condition (vii) of Theorem 3.2, we have the following induction scheme where we induct over the open sets of a well-quasi-order. Note that $\emptyset$ is the smallest open set with respect to inclusion.

Theorem 3.4. *Suppose $(Q, \preceq)$ is a well-quasi-order, and let φ be a statement for which:*

1. *$\varphi(\emptyset)$.*

2. *For all open subsets $X \subseteq Q$, we have $\varphi(X)$ is provided φ holds for all downward closed proper subsets $Y \subset X$.*

Then φ holds for the set Q.

Proof. Suppose 1 and 2 were satisfied, and yet φ did not hold for Q. By condition 2, there would exist a proper open subset $Q_1 \subset Q$ for which φ did not hold. By condition 1, $Q_1 \neq \emptyset$. So there exists a proper open subset $Q_2 \subset Q_1$ for which φ does not hold. Continuing the argument, we would have a strictly decreasing chain of open sets $Q \supset Q_1 \supset Q_2 \supset \cdots$ which is a contradiction. Thus, the induction scheme is valid. $\square$

Definition. *Let $(Q_1, \preceq_1), (Q_2, \preceq_2)$ be quasi-orders. An **order homomorphism** is a function $f : Q_1 \to Q_2$ such that $\forall x_1, x_2 \in Q_1 (x_1 \preceq x_2 \Rightarrow f(x_1) \preceq f(x_2)$.*

Definition. *Let $(P_1, \preceq)$ be a partial order. $(P_2, \preceq_2)$ is called **linear extension** of P_1 if P_2 is total order and there exists a bijective order homomorphism $f : P_1 \to P_2$. We say that $\preceq_2$ extends $\preceq_1$.*

Definition. *A **well-partial-order** is a partial order which is also a well-quasi-order.*

Proposition 3.5. *Every linear extension of a well partial order is a well order.*

Proof. Let $(Q_2, \preceq_2)$ be a linear extension of the well partial order $(Q_1, \preceq_1)$. Since Q_2 is a total order, it suffices to show that Q_2 contains no infinite strictly decreasing sequences. Since Q_2 is a linear extension of Q_1, there exists a bijective order homomorphism $f : Q_1 \to Q_2$. Let $x_1, x_2, x_3, ...$ be an arbitrary sequence of elements from Q_2. Then $f^{-1}(x_1), f^{-1}(x_2), f^{-1}(x_3), ...$ is a sequence of elements from Q_1. Since Q_1 is a well-quasi-order, there exists $i < j$ such that $f^{-1}(x_i) \preceq_1 f^{-1}(x_j)$. Note that f is an order homomorphism, so $x_i = f(f^{-1}(x_i)) \preceq_2 f(f^{-1}(x_j)) = x_j$. Therefore, $x_i \preceq x_j$, which shows that $x_1, x_2, x_3, ...$ can't be strictly decreasing. Since $x_1, x_2, x_3, ...$ was arbitrary, Q_2 has no infinite strictly decreasing sequences. $\square$

4 DICKSON'S LEMMA

Given n quasi-orders, $(Q_1, \preceq_1)$,$(Q_2, \preceq_2)$,. . .,$(Q_n, \preceq_n)$, we can form the product quasi-order $(Q, \preceq^n)$ on the product set $Q = Q_1 \times Q_2 \times \cdots \times Q_n$ in the following way. Let $a_1, b_1 \in Q_1$,$a_2, b_2 \in Q_2$,. . ., $a_n, b_n \in Q_n$. Then we say $(a_1, a_2, \ldots, a_n) \preceq^n (b_1, b_2, \ldots, b_n)$ provided $a_1 \preceq_1 b_1$,$a_2 \preceq_2 b_2$,. . .,$a_n \preceq_n b_2$. This defines a quasi-order on Q since reflexitivity and transitivity hold component wise. The following was proven in [Dickson, 1913]

Theorem 4.1. *(Dickson's Lemma) Let $\mathbb{N}^n$ be quasi-ordered by $\leq^n$. Then any nonempty subset of $\mathbb{N}^n$ contains finitely many minimal elements.*

Given the equivalent conditions given by Theorem 3.2 for when a quasi-order is a well-quasi-order, Dickson's Lemma is equivalent to saying $(\mathbb{N}^n, \leq^n)$ is a well-quasi-order. Dickson's Lemma follows from the following more general theorem.

Theorem 4.2. *(Product Theorem for Well-Quasi-Orders) Let $Q_1, Q_2, \ldots, Q_n$ be well-quasi-ordered by $\preceq_1, \preceq_2, \ldots, \preceq_n$ respectively. Then $Q = Q_1 \times Q_2 \times \cdots \times Q_n$ is well-quasi-ordered by $\preceq^n$.*

Proof. We will prove Q is a well-quasi-order by using the increasing subsequence condition. That is, given a sequence of elements from Q, we will show it has an increasing subsequence. So let $A_0 \subseteq Q$, be a sequence of n-tuples. Then A_0 has a subsequence, A_1, that increases in the first component since Q_1 is well-quasi-ordered. Also A_1 has a subsequence, A_2 that increases in the second component since Q_2 is well-quasi-ordered. Note that A_2 is increasing in the first component also since A_2 is a subsequence of A_1. If we continue taking increasing subsequences in this manner, then A_i will be a sequence which is increasing in the first i components, and A_n will be a sequence in which all components are increasing. Thus, every sequence of elements from Q has an increasing subsequence. This shows that Q is a well-quasi-order. $\square$

4.1 GRÖBNER BASES AND BUCHBERGER'S ALGORITHM

Here we present an application of Dickson's Lemma to the theory of Gröbner bases. A Gröbner basis is a basis for an ideal which satisfies certain nice properties. For example, consider the following problem:

Given $f \in R = k[x_1, x_2, \ldots, x_n]$ and $B = \{b_1, b_2, \ldots, b_n\} \subseteq R$, is $f \in \langle B \rangle$?

This problem is easy if B is a Gröbner basis. Fortunately, if B is not a Gröbner basis, then there is an algorithm called Buchberger's algorithm which transforms B into a finite Gröbner basis B' which generates the same ideal, i.e. $\langle B \rangle = \langle B' \rangle$. So the problem becomes easy when determining whether $f \in \langle B' \rangle = \langle B \rangle$. Both the ideas of a Gröbner bases and Buchberger's algorithm were developed in Bruno Buchberger's 1965 P.h.D thesis [Buchberger, 2006]. Gröbner bases are named after Buchberger's P.h.D. thesis advisor, Wolfgang Gröbner. We are interested in Buchberger's algorithm because the proof that it terminates relies on Dickson's Lemma.

We will consider the ring $R = k[x_1, x_2, \ldots, x_n]$. A monomial in R is an element of the form $x_1^{m_1} x_2^{m_2} \cdots x_n^{m_n}$. For $\alpha = (a_1, a_2, \ldots, a_n) \in \mathbb{N}^n$ we use x^α to denote the monomial $x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n}$. It is clear that x^α divides x^β if and only if $\alpha \leq^n \beta$. Using this notation, every element of R can be written in the form $k_1 x^{\alpha_1} + k_2 x^{\alpha_2} + \cdots + k_n x^{\alpha_n}$.

Definition. *A subset $I \subseteq R$ is an **ideal** of R if it satisfies $\forall x, y \in I$, we have $x + y \in I$ and $\forall x \in I, \forall a \in R$, we have $ax \in R$.*

Definition. *Let $f \in R$ and $B \subseteq R$. We say f is a **poly-linear combination** of elements of B if there exists $b_1, b_2, \ldots, b_n \in B$ and $h_1, h_2, \ldots, h_n \in R$ such that $f = h_1 b_1 + h_2 b_2 + \cdots + h_n b_n$.*

Definition. *The set generated by B is $\{f \in R |\ f$ is a poly-linear combination of elements of $B\}$ denoted by $\langle B \rangle$.*

Definition. *Let $I \subseteq R$ be an ideal, and $B \subseteq R$. We say B a* ***basis*** *of I if $\langle B \rangle = I$.*

Note that B is a basis of the ideal $\langle B \rangle$.

A term ordering, $\preceq$, on R is a total ordering of the monomials of R that satisfies $\forall \alpha, \beta, \gamma, \in \mathbb{N}^n$:

1. $1 \preceq x^\alpha$

2. $x^\alpha \preceq x^\beta \Rightarrow x^\alpha x^\gamma \preceq x^\beta x^\gamma$.

Proposition 4.3. *Any term ordering on R extends the divisibility ordering of the monomials of R.*

Proof. Let x^α divide x^β, with $\alpha = (a_1, a_2, \ldots, a_n)$ and $\beta = (b_1, b_2, \ldots, b_n)$. For $\gamma = (b_1 - a_1, b_2 - a_2, \ldots, b_n - a_n)$ We have $1 \preceq x^\gamma$ by property 1. and $x^\alpha \preceq x^\alpha x^\gamma = x^\beta$ by property 2. □

Proposition 4.4. *Any term ordering, $\preceq$, on R is a well ordering of the monomials of R.*

Proof. By the previous proposition, any term ordering is a linear extension of the divisibility ordering of monomials. The divisibility ordering of monomials is a well-quasi-order, since it is isomorphic to $(\mathbb{N}^n, \leq^n)$. Thus, any term ordering is a linear extension of a well-quasi-order, and is therefore a well order. □

When R is equipped with a term order $\preceq$, we can write any $f \in R$ in decreasing term order, i.e. $f = k_1 x^{\alpha_1} + k_2 x^{\alpha_2} + \cdots + k_n x^{\alpha_n}$, where $x^{\alpha_1} \succ x^{\alpha_2} \succ \cdots \succ x^{\alpha_n}$. We call x^{α_1} the leading term of f and denote it by $\mathrm{interm}(f)$, while k_1 is the leading coefficient of f denoted by $\mathrm{incoef}(f)$. In what follows, we will assume that R has a fixed term order.

Definition. *Let $B \subseteq R$, and let $f_1, f_2 \in R$. We say that f_2 is a* ***one-step reduction*** *of f_1, by B, and write $f_1 \rightarrow_B f_2$ if there exists $g \in B$ such that* $\mathrm{interm}(g)$ *divides* $\mathrm{interm}(f_1)$

and

$$f_2 = f_1 - \frac{g\,\mathrm{incoef}(f_1)\,\mathrm{interm}(f_1)}{\mathrm{incoef}(g)\,\mathrm{interm}(g)}.$$

If there exists a finite sequence of one step reductions $f_1 \longrightarrow_B f_2 \longrightarrow_B \cdots \longrightarrow_B f_n$*, then we write* $f_1 \twoheadrightarrow_B f_n$ *and say* f_1 ***reduces*** *to* f_n*. Note that we allow the trivial case,* $f_1 \twoheadrightarrow_B f_1$*. If* f_1 *reduces to* f_n *and there does not exist* f_{n+1} *such that* $f_n \longrightarrow_B f_{n+1}$ *then we say that* f_1 ***completely reduces*** *to* f_n*. We also say that* f_n *is* ***completely reduced****.*

Proposition 4.5. *There are no infinite reduction sequences* $f_1 \longrightarrow_B f_2 \longrightarrow_B f_3 \ldots$*.*

Proof. If there were an infinite reduction sequence, then $\mathrm{interm}(f_1) \succ \mathrm{interm}(f_2) \succ \mathrm{interm}(f_3) \ldots$ would be an infinite strictly decreasing sequence of monomials of R. However, $\preceq$ well orders the monomials which is a contradiction. □

Proposition 4.6. *Every polynomial* $f \in k[x_1, x_2, \ldots, x_n]$ *can be reduced to a completely reduced polynomial* h*.*

Proof. We can reduce f until it is completely reduced. If not, we would have an infinite reduction sequence contrary to the above proposition. □

It is obvious that $f \twoheadrightarrow_B 0 \Rightarrow f \in \langle B \rangle$. However, it may be the case that $f \in \langle B \rangle$ and yet f can't be reduced to 0. This is the motivation for the following definition.

Definition. *Let* I *be an ideal of* R *and* $B \subseteq R$*. Then* B *is a* ***Gröbner basis*** *of* I *if* B *is a basis of* I *and* $f \in I \Rightarrow f \twoheadrightarrow_B 0$*.*

It is obvious that any ideal I of R has a Gröbner basis since I is a Gröbner basis of I. However, by Dickson's Lemma we can say something a lot stronger.

Theorem 4.7. *Every ideal* $I \subseteq R$ *has a finite Gröbner basis.*

Proof. Consider the set of monomials $A = \{\text{interm}(f) | f \in I\}$. By Dickson's Lemma, there exists a finite set $M \subseteq A$ of elements which are minimal with respect to the divisibility ordering of monomials. For each $m \in M$ choose an $x_m \in I$ which has initial term m. Let $B = \{x_m | m \in M\}$. We will show that B is a Gröbner basis of I. Let $f \in I$ be non-zero. Then $\text{interm}(f) \in A$ is divisible by some $m \in M$. Since $\text{interm}(x_m) = m$ divides $\text{interm}(f)$, we have that f can be reduced by x_m. So every element of I is either 0 can be reduced by some element of B. Thus, $f \in I$ implies $f \twoheadrightarrow_B 0$. □

The amazing thing about Buchberger's algorithm is that for a finite set, $B \subset R$, it gives us a method for fining a finite Gröbner basis of $\langle B \rangle$. The basic idea behind Buchberger's algorithm is to take pairs of elements of B, try to cancel their initial terms and then reduce. If the reduced term is not 0, then expand the set B by including the reduced term.

First we need to define the canceling operation. This is the S-polynomial of f and g which is the polynomial formed from canceling a multiple of the initial term of f with a multiple of the initial term of g.

Definition. *Let* $f, g \in R$ *with* $\text{incoef}(f) = x_1^{\alpha_1} x_2^{\alpha_2} \dots x_n^{\alpha_n}$, $\text{incoef}(g) = x_1^{\beta_1} x_2^{\beta_2} \dots x_n^{\beta_n}$. *The* ***S-polynomial*** *of* f *and* g *denoted by* $S(f, g)$ *is*

$$\frac{fx^{\gamma-\alpha}}{\text{incoef}(f)} - \frac{gx^{\gamma-\beta}}{\text{incoef}(g)}$$

where $\gamma = \gamma_1, \gamma_2, \dots, \gamma_n$ *for* $\gamma_1 = \max(\alpha_1, \beta_1), \gamma_2 = \max(\alpha_2, \beta_2), \dots, \gamma_n = \max(\alpha_n, \beta_n)$.

We can now precisely state Buchberger's algorithm, which takes as input a finite set B and outputs a finite Gröbner basis for the ideal $\langle B \rangle$.

The algorithm runs as follows:

1. Let $S = \{S(b_i, b_j) | b_i, b_j \in B \text{ with } i < j\}$

2. Reduce the elements of S by B until they are completely reduced. Call this new set S'.

3. If $S' = 0$, then B is a Gröbner basis and we are done.

4. Else repeat the algorithm with $B \cup S'$ in place of B.

Theorem 4.8. *Buchberger's algorithm always terminates.*

Proof. Assume Buchberger's algorithm did not halt. Then $S' \neq 0$ for each iteration of the algorithm. Let $r_i \neq 0$ be an element of S' after the i^{th} iteration. Consider the sequence {interm(r_1),interm(r_2),...}. Since the monomials are well-quasi-ordered by the divisibility relation, $\exists i, j \in \mathbb{N}$ such that r_i divides r_j. This means r_j can be reduced by r_i. However, in the j^{th} iteration of the program, $r_i \in B$ and r_j is in a completely reduced form. This is a contradiction. $\square$

Buchberger's algorithm always outputs a Gröbner basis as a consequence of the following non-trivial theorem.

Theorem 4.9. *(Buchberger's criterion) A basis B is a Gröbner basis if and only if $\forall b_1, b_2 \in B$ we have $S(b_1, b_2) \twoheadrightarrow_B 0$.*

We note that the results of this section give us a simple method for deciding whether a polynomial f is in $\langle B \rangle$ where B is a finite set of polynomials. To do this simply use Buchberger's algorithm to transform B into a Gröbner basis B'. If $f \twoheadrightarrow_{B'} 0$ then $f \in \langle B \rangle$, otherwise $f \notin \langle B \rangle$.

5 HIGMAN'S LEMMA

Let Q be quasi-ordered by $\preceq$. We use Q^* to denote the set of finite strings with symbols from Q. Also, we shall use ϵ to denote the empty string. We can quasi-order Q^* by the following relation: Let $s_1, s_2 \in Q^*$, and let m and n be the length of s_1 and s_2 respectively. Then we define $s_1 \preceq^* s_2$ if there exists an increasing function $f : [m] \to [n]$ such that the i^{th} symbol of s_1 is $\preceq$ the $f(i)^{th}$ symbol of s_2. We say that f is an embedding of s_1 into s_2. It is obvious that $\preceq^*$ quasi-orders Q^*. However, for Q^* to be a well-quasi-order, it is necessary that Q is also a well-quasi-order since the set of strings of length 1 ordered by $\preceq^*$ are order isomorphic to $(Q, \preceq)$. The content of the next theorem is that for Q^* to be a well-quasi-order, it is sufficient that Q is a well-quasi-order.

Theorem 5.1. *(Higman's Lemma) If Q is a well-quasi-order, then Q^* is a well-quasi-order.*

Higman's Lemma was originally proven in [Higman, 1952] to prove that certain algebras are finitely generated. Here we will present an original proof of Higman's Lemma which uses both the Product Theorem for well-quasi-orders and the induction scheme for open subsets of a well-quasi-order.

Proof. Let Q be a well-quasi-order. For the open subsets $X \subseteq Q$, Let $\varphi(X)$ be the statement:

"X^* is a well-quasi-order". We will show that this statement satisfies the two necessary properties for induction over open sets.

1. We need that φ holds for $\emptyset$. This is true because $\emptyset^* = \{\epsilon\}$ and $\epsilon \preceq^* \epsilon$.

2. Let $X \subseteq Q*$ be open. We must show that φ holds for X assuming φ holds for all proper open subsets of X. This means showing X^* is a well-quasi-order when assuming Y^* is a well-quasi-order for all open $Y \subseteq X$. To do this we will use the increasing subsequence characterization of well-quasi-orders. That is, we will take an arbitrary

sequence from X^* and show that it has an increasing subsequence. So let $\{a_n\} \subseteq X^*$ be an arbitrary sequence. If for any a_i there exists a_j such that $a_i \preceq^* a_j$ and $i < j$, then we can construct an increasing subsequence starting with a_1 and subsequently taking increasing elements with greater indices. However, if there exists an a_i such that $i < j$ implies we don't have $a_i \preceq^* a_j$ we can do the following. Let m be the length of a_i, let $x = a_i$ and for $k \leq m$ let x_k denote the k^{th} symbols of x. We can define a function $f : \{a_j | i < j\} \to \{0, 1, \ldots, m-1\}$ where $f(a_j) = \max\{k \in \mathbb{N} | x_1 x_2 \ldots x_k \preceq^* a_j\}$. By the pigeonhole principle, there exists an infinite subsequence $\{b_n\} \subseteq \{a_n\}$ such that f is constant on $\{b_n\}$. Let $l = f(\{b_n\})$. By definition of f, this means $x_1 x_2 \ldots x_l \preceq^* b_i$, yet we don't have $x_1 x_2 \ldots x_l x_{l+1} \preceq^* b_i$ for all $i \in \mathbb{N}$. We will use the fact that $x_1 x_2 \ldots x_l \preceq^* b_i$ to partition b_i into $2l + 1$ substrings. Let g be an embedding of $x_1 x_2 \ldots x_l$ into b_i using the greedy approach. This means g embeds x_1 into the first possible symbol of b_i, it embeds x_2 into the next possible and so on. So b_i is the concatenation of $2l + 1$ substrings: $s_1, g(x_1), s_2, g(x_2), s_3, \ldots, s_l, g(x_l), s_{l+1}$. Note that because g is the greedy embedding, x_j does not embed into s_j for $1 \leq j \leq l$. Also, since $x_1 x_2 \ldots x_l x_{l+1} \not\preceq b_i$, we have that x_{l+1} does not embed into s_{l+1}. So $(s_1, g(x_1), s_2, g(x_2), \ldots, s_l, g(x_l), s_{l+1}) \in Y_1^* \times Q \times Y_2^* \times Q \times \cdots \times Y_l^* \times Q \times Y_{l+1}^*$, where $Y_j = X - \mathrm{cl}(x_j)$. Each Y_j is an open proper subset of X, so by the induction hypothesis each Y_j is well-quasi-ordered. Since Q is also a well-quasi-order, the product $Y_1^* \times Q \times Y_2^* \times Q \times \cdots \times Y_l^* \times Q \times Y_{l+1}^*$ is well-quasi-ordered. The sequence $\{b_n\}$ has a subsequence $\{c_n\}$ that is increasing with respect to the product ordering on $Y_1^* \times Q \times Y_2^* \times Q \ldots Y_l^* \times Q \times Y_{l+1}^*$. It is apparent that $\{c_n\}$ increases with respect to string embedding as well. We showed $\{a_n\}$ has an increasing subsequence, $\{c_n\}$. Thus, X is a well-quasi-order, which means φ holds for X. $\square$

6 KRUSKAL'S TREE THEOREM

Kruskal's Tree Theorem is a generalization of Higman's Lemma that was first proven in [Kruskal, 1960].

Let $(Q, \preceq)$ be a quasi-order. We will consider the set Q^T of all finite rooted trees with labels from Q. That is, for a tree $T \in Q^T$, T has assigned to each of its vertices an element of Q. For $T \in Q^T$ we let $V(T)$ denote the vertices of T. For $T_1, T_2 \in Q^T$ we say T_1 embeds into T_2 and write $T_1 \preceq^T T_2$ if there exists an injective function $f : V(T_1) \to V(T_2)$ such that $\forall v \in V(T_1)$, the ancestors of v are mapped to the ancestors of $f(v)$. It is simple to show that Q^T is a quasi-order. However, if Q is well-quasi-ordered, then more can be said.

Theorem 6.1. *(Kruskal's Tree Theorem) Q^T is well-quasi-ordered by $\preceq^T$ provided Q is well-quasi-ordered by $\preceq$.*

One interesting application of Kruskal's Tree Theorem is in reverse mathematics, a program of mathematical logic, which was founded by Harvey Friedman in 1974. The goal of reverse mathematics is to classify mathematical statements by the logical systems needed to prove them. To do this, one assumes the mathematical statement in question and uses it to prove the induction scheme associated with the given axiom logical system over a base system. This is a reversal of the "usual" goal of mathematics which is to prove statements based on assumed axioms. Friedman's finite form is a modification of Kruskal's Tree Theorem which can be stated in first order arithmetic yet can only be proven in a relatively strong system of second order arithmetic[Simpson, 1985].

Theorem 6.2. *(Friedman's Finite Form)For any $k \in \mathbb{N}$ There exists an m large enough such that for every finite sequence of rooted (non-labeled) trees $T_1, T_2, ..., T_m$ with $|V(T_i)| \leq i + k$ for $1 \leq i \leq m$, there exists indices i, j such that $1 \leq i < j \leq m$ and $T_i \preceq^T T_j$.*

Here we show how Friedman's finite form follows from both Kruskal's tree theorem and König's Lemma.

Proof. Assume k is fixed. We will define a tree, A, which will have as vertices all finite sequences of trees $T_1, T_2, \ldots, T_n$ which satisfy $|V(T_i)| < i + k$ for $1 \leq i \leq n$ and $\nexists i, j$ such that $i < j$ and $T_i \preceq^T T_j$. Let the root of A be the empty sequence, and the children of the root be trees T_1 such that $|V(T_1)| < 1 + k$. In general, assume the n^{th} level of A is defined and has vertices of the form $T_1, T_2, \ldots, T_n$. Then for each given vertex, $T_1, T_2, \ldots, T_n$, let its children be sequences which extend $T_1, T_2, \ldots, T_n$ by some T_{n+1} such that $|V(T_{n+1})| < n + 1 + k$ and $\forall i < n + 1, T_i \npreceq^T T_{n+1}$. Note that each vertex of A has only finitely many children since there are only finitely many trees with less than $n+1+k$ vertices. By König's Lemma, if A has paths of arbitrarily long finite length, then A has an infinite path. However, an infinite path of A would correspond to an infinite sequence of trees $T_1, T_2, T_3, \ldots$ such that $\nexists i, j$ with $i < j$ and $T_i \preceq^T T_j$. This would contradict Kruskal's Tree Theorem. So A has no infinite paths and thus, A does not have paths of arbitrarily long finite length. Therefore, the lengths of paths in A are bounded by some $m - 1$. Let $S = T_1, T_2, \ldots, T_m$, satisfy $|V(T_i)| < i + k$ for $1 \leq i \leq m$. Then $S \notin A$ since S is a sequence of length m and A only contains sequences of length less than m. Since $S \notin A$, and yet satisfies $|V(T_i)| < i + k$ for $1 \leq i \leq m$, there exists i, j such that $1 \leq i \leq j \leq m$ and $T_i \preceq^T T_j$. $\square$

In the proof of Friedman's finite form, we made use of König's Lemma. This is, in a sense, necessary in that Friedman's finite form corresponds to a subsystem of second order arithmetic which is stronger than the system WKL_0 which expressly takes König's Lemma to be an axiom.

7 THE ROBERTSON-SEYMOUR THEOREM

The Robertson-Seymour Theorem (also known as the graph minor theorem) was proven in [Robertson and Seymour, 2004] at the culmination of twenty papers written as part of the Graph Minors Project. The Robertson-Seymour Theorem essentially states that the set of finite graphs is well-quasi-ordered by the graph minor relation.

Definition. *A graph H is a **minor** of a graph G if H can be obtained from G by a finite sequence of contractions and deletions. A set S of graphs is **minor-closed** if $\forall G \in S$ we have H is a minor of G implies $H \in S$.*

It is easy to show that the set of finite graphs is quasi-ordered by the minor relation. However, more can be said.

Theorem 7.1. *(The Robertson-Seymour Theorem) Every minor-closed family of graphs can be characterized by a finite set of excluded minors.*

By condition (v) of Theorem 3.2, The Robertson-Seymour is equivalent to saying that the set of finite graphs is well-quasi-ordered. It is interesting to note that condition (v) applies to all of the previous well-quasi-orders that were studied. However, condition (v) appears to be more important in this context in that there are many interesting minor-closed families of graphs.

One of the first theorems to give a characterization of a minor-closed graphs in terms of excluded minors is Kuratowski's Theorem.

Theorem 7.2. *(Kuratowski's Theorem) A finite graph is planar if and only if it contains neither K_5 nor $K_{3,3}$ as a minor.*

That the set of excluded minors in Kuratowki's Theorem is finite follows from the Robertson-Seymour Theorem, since the set of planar graphs is minor-closed. In general, the set of graphs which embed in a given surface is minor-closed. So the following result can be seen as a corollary of the Robertson-Seymour Theorem.

Theorem 7.3. *The set of graphs which embed in a given surface can be characterized by a finite set of excluded minors.*

There are many other interesting families of graphs that are minor-closed. For instance the set of series-parallel graphs is minor-closed. The following characterization of series-parallel graphs was given by Dirac in [Dirac, 1952]:

Theorem 7.4. *A graph is series-parallel if and only if it contains no K_4 minor.*

While minor-closed sets of graphs are guaranteed such a finite characterization, it is in general difficult to determine what these characterizations are.

The work of Robertson and Seymour led to many structural results for graphs and as a result, Robertson and Seymour proved the following amazing algorithmic result.

Theorem 7.5. *For a given graph H, there exists a polynomial time algorithm for deciding whether a graph contains H as a minor.*

As a consequence of both the Robertson-Seymour Theorem and Theorem 7.5 the following remarkable theorem holds.

Theorem 7.6. *Any minor-closed property of graphs can be tested in polynomial time.*

This is because any minor-closed family F has a finite set of excluded minors $\{H_1, H_2, \ldots, H_n\}$, and testing whether a graph G is in F amounts to testing whether none of $H_1, H_2, \ldots, H_n$ are a minor of G. So testing whether G is in F can be done in polynomial time, since there is a polynomial time algorithm for testing whether H_i is in G for $1 \leq i \leq n$.

Of course, if the finite set of excluded minors is not known, it may be difficult to actually determine a polynomial time algorithm for membership.

8 WELL-QUASI-ORDERING CONJECTURE FOR F-REPRESENTABLE MATROIDS

Following the success of the Graph Minors Project, work is now being done on a Matroid Minors Project. The goal of this project is to find structural results for matroid minors and to prove a theorem for $\mathbb{F}$-representable matroids which is analogous to the Robertson-Seymour Theorem.

Definition. *A **matroid** is a pair $M = (E, B)$, with E a finite set called the ground set, and B a subset of the power set of E which satisfies:*

1. $B \neq \emptyset$
2. $\forall B_1, B_2 \in B$ *we have* $\forall a_1 \in B_1 - B_2, \exists a_2 \in B_2 - B_1$ *such that* $(B_1 \cup \{a_2\}) - \{a_1\} \in B$.

Definition. *Let M be a matroid with ground set E and with B the collection of base elements. The **dual** matroid of M written as M^* is the matroid which has ground set E yet with collection of base elements $B^* = \{E - B_i | B_i \in B\}$*

Like graphs, matroids also have contraction and deletion operations.

Definition. *For a matroid $M = (E, B)$ with $R \subseteq E$, the matroid $M \backslash R$ is the **deletion** of R from M. The matroid $M \backslash R$ has as ground set $E - R$ and collection of base elements $\{D | D \subseteq B_i \cap (E - R)$ for some $B_i \in B$ and D is a maximal set with this property$\}$*

Definition. *For a matroid $M = (E, B)$ with $R \subseteq E$, the matroid $(M^* \backslash R)^*$ is called the **contraction** of R in M and is denoted by M/R.*

Note that contraction and deletion are dual operations in that $M/R = (M^* \backslash R)^*$ and $M \backslash R = (M^*/R)^*$

Definition. *Let M_1 and M_2 be matroids. M_1 is a **minor** of M_2 if M_1 can be formed from M_2 by a finite sequence of contractions and deletions. A set S of matroids is **minor-closed** if $\forall M \in S$ we have N is a minor of M implies $N \in S$.*

The set of all matroids is quasi-ordered by the matroid minor relation, yet it is known that this set is not well-quasi-ordered. However, it is suspected that a particular class of matroids, the $\mathbb{F}$-representable matroids, *are* well-quasi-ordered.

Definition. *For a finite field $\mathbb{F}$ an **$\mathbb{F}$-representable matroid** is a matroid which has as ground set a set of vectors $E = \{v_1, v_2, \ldots, v_k\} \subseteq \mathbb{F}^n$ and which has for the collection of base elements, the maximal linearly independent subsets of E.*

As a consequence of the following theorem, it makes sense to consider $\mathbb{F}$-representable matroids as being quasi-ordered by the matroid minor relation.

Theorem 8.1. *Any matroid minor of an $\mathbb{F}$-representable matroid is also an $\mathbb{F}$-representable matroid.*

Geelen, Gerards, and Whittle in [Geelen et al., 2007] make the following two conjectures which can be seen as the matroid analogs of the Robertson-Seymour Theorem and Theorem **??** respectively.

Conjecture 8.2. *(Well-Quasi-Ordering Conjecture for Matroids) For any finite field $\mathbb{F}$, the $\mathbb{F}$-representable matroids are well-quasi-ordered by the matroid minor relation.*

Conjecture 8.3. *For any finite field $\mathbb{F}$, and $\mathbb{F}$-representable matroid N, there exists a polynomial time algorithm for deciding whether an $\mathbb{F}$-representable matroid contains an N minor.*

Similar to the situation for graphs, Conjecture 8.2 and Conjecture 8.3 would imply the following.

Conjecture 8.4. *For any finite field $\mathbb{F}$, any minor-closed property of $\mathbb{F}$-representable matroids can be tested in polynomial time.*

REFERENCES

B. Buchberger. An algorithm for finding the basis elements of the residue class ring of a zero dimensional polynomial ideal. *J. Symbolic Comput.*, 41(3-4): 475–511, 2006. Translated from the 1965 German original by Michael P. Abramson.

L. E. Dickson. Finiteness of the Odd Perfect and Primitive Abundant Numbers with n Distinct Prime Factors. *Amer. J. Math.*, 35(4):413–422, 1913.

G. A. Dirac. A property of 4-chromatic graphs and some remarks on critical graphs. *J. London Math. Soc.*, 27:85–92, 1952.

J. Geelen, B. Gerards, and G. Whittle. Towards a matroid-minor structure theory. In *Combinatorics, complexity, and chance*, volume 34 of *Oxford Lecture Ser. Math. Appl.*, pages 72–82. Oxford Univ. Press, Oxford, 2007.

G. Higman. Ordering by divisibility in abstract algebras. *Proc. London Math. Soc. (3)*, 2:326–336, 1952.

J. B. Kruskal. Well-quasi-ordering, the Tree Theorem, and Vazsonyi's conjecture. *Trans. Amer. Math. Soc.*, 95:210–225, 1960.

N. Robertson and P. D. Seymour. Graph minors. XX. Wagner's conjecture. *J. Combin. Theory Ser. B*, 92(2):325–357, 2004.

S. G. Simpson. Nonprovability of certain combinatorial properties of finite trees. In *Harvey Friedman's research on the foundations of mathematics*, volume 117 of *Stud. Logic Found. Math.*, pages 87–117. North-Holland, Amsterdam, 1985.